NONLINEAR WAVES ON WATER AND THEORY OF SOLITONS

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As a basis for a mathematical model to describe wave propagation in different media, use is commonly made of the wave equation, which in the one-dimensional approximation is as follows [1]:

$$u_{tt} = c_0^2 u_{xx} \,. \tag{1}$$

The wave characteristic u in this equation depends on the space coordinate x and the time t.

Equation (1) describes a plane one-dimensional wave, an analog of which is the wave in a string. As the characteristic $u(x\ t)$ in (1) the air density is adopted, if one is concerned, for instance, with a sound wave in air. If electromagnetic waves are under consideration, then by $u(x, t)$ the intensity of an electric or magnetic field is understood, and so on.

The solution of the Cauchy problem for wave equation (1), which was first obtained by d'Alembert in 1748, has the form

$$u(x, t) = f(x - c_0 t) + q(x + c_0 t) \,. \tag{2}$$

Here, the functions f and q are found from the initial conditions for $u(x, t)$. Since Eq. (1) contains the second derivative of u with respect to t, two initial conditions are set for it: the value of u at $t = 0$ and the value of the derivative u_t at $t = 0$.

Wave equation (1) has an important property that is essentially as follows. It is obvious that if any two solutions of it are taken, their sum will again be a solution of this equation. This special feature represents the principle of superposition of solutions of (1) and corresponds to linearity of the phenomenon that it describes. For nonlinear models, it is not fulfilled, which leads to substantial differences in the processes that take place in the corresponding models.

It is not always easy to determine with what (linear or nonlinear) model an investigator is concerned, but when a mathematical model is formulated, this is easier to do and fulfillment of the principle of superposition of solutions can be checked.

Passing to waves on water, we note that they can be analyzed using the well-known equations of hydrodynamics, which are known to be nonlinear. Therefore in the general case, waves on water are nonlinear. They can be considered to be linear only in the limiting case of small amplitudes.

Discovery of a Solitary Wave and the Korteveg–de Vries Equation. Despite the fact that nonlinear waves are often encountered in nature they were discovered only in 1834. This discovery (like many others) was made by chance when Russell, an English shipbuilder, carried out a test of a barge on one of the canals near Edinburg. This event is a well-known fact since it is described in detail by the investigator himself [2-4]. Russell was engaged in investigation of movement of the barge along the canal, which was pulled by a pair of horses. Suddenly the barge stopped but the mass of water brought into motion by it concentrated at the barge's bow and then separated from it. Next, this mass of water rolled along the canal at a high velocity in the form of a solitary hump without changing its shape or decreasing its velocity.

Throughout his life Russell repeatedly returned to observation of this wave since he believed that the solitary wave discovered by him played an important role in many phenomena in nature. He established some

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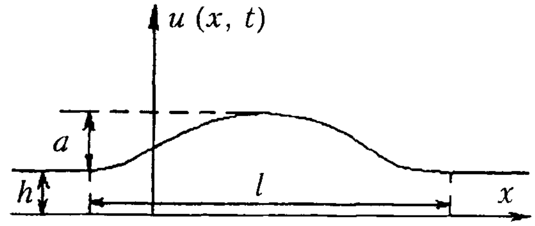

Fig. 1. Schematic of a solitary wave propagating along a canal and its parameters.

properties of this wave: first, he noted that it moved with a constant velocity without changing its shape; second, he found the dependence of the wave velocity on the channel depth h and the wave height a:

$$C = \sqrt{g\,(a + h)}\ ; \tag{3}$$

third, he found that one large wave can disintegrate into several waves; fourth, he noted that in the experiments only humped waves were observed. Once he also saw that the solitary waves discovered passed through each other without any changes. However, he did not pay serious attention to this very important property.

The work published by Russell in 1844 aroused a negative reaction among the scientific community. In Great Britain, Airy and Stokes took interest in it; however, they cast doubt on the results of Russell's observations. Airy noted that the theory of long waves on shallow water fails to confirm Russell's conclusions and asserted that long waves cannot retain their form unchanged. Stokes, one of the founders of modern hydrodynamics, also regarded the fact of the existence of a solitary wave critically.

After such negative treatment of the solitary-wave discovery, the latter was forgotten for a long time. The correctness of Russell's observations was confirmed later by Boussinesq (1872) and Rayleigh (1876), who independently of one another derived a formula for a hump of the free surface on water in the form of a squared hyperbolic secant and calculated the velocity of solitary-wave propagation on water.

The problem that had arisen as a result of Russell's experiments was finally clarified owing to the work of the Danish scientists Korteveg and de Vries, who generalized the Rayleigh method and in 1895 derived an equation to describe long waves on water. Using the equations of hydrodynamics, Korteveg and de Vries considered the deviation $u(x, t)$ of a water surface from the position of equilibrium in the absence of vortices and at constancy of the water density. They also assumed that two conditions were satisfied for dimensionless parameters (see Fig. 1):

$$\varepsilon = \frac{a}{h} << 1\ ,\quad \delta = \frac{h}{l} << 1\ . \tag{4}$$

Essentially, the approximations were based on the fact that the amplitude of the waves under consideration was much less than the depth of the body of water [5].

The equation obtained by Korteveg and de Vries in canonical form is as follows:

$$u_t + 6uu_x + u_{xxx} = 0\ . \tag{5}$$

Equation (5) has a solution in the variables of a running wave, known since the end of the last century and expressed in terms of the Jacobi elliptical function:

$$u\,(x, t) = \beta + (\alpha - \beta)\, cn^2 \left\{ \sqrt{\left(\frac{\alpha - \gamma}{2}\right)}\ (\xi - \xi_0)\ ,\ s^2 \right\},\quad s^2 = \frac{\beta - \gamma}{\alpha - \gamma}\ , \tag{6}$$

where $\xi = x - c_0 t$ and α, β, γ $(\alpha \geq \beta \geq \gamma)$ are real roots of the cubic equation

$$U^3 - \frac{C_0 U^2}{2} + C_1 U + C_2 = 0\ . \tag{7}$$

The solution (6) is a wave with the period

$$T = 2\sqrt{\left(\frac{2}{\alpha - \gamma}\right)} \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - s^2 x)}} = \sqrt{\left(\frac{8}{\alpha - \gamma}\right)} K(s) . \tag{8}$$

If $\alpha > \beta = \gamma$, then period (8) becomes infinite. In this case, from (6) a solitary wave is obtained:

$$u(x, t) = \beta + (\alpha - \beta)\,\mathrm{ch}^{-2}\left\{\sqrt{\left(\frac{\alpha - \gamma}{2}\right)}(\xi - \xi_0)\right\} . \tag{9}$$

Commonly it is assumed that $\beta = \gamma = 0$, $\alpha = 2k^2$, and then solution (9) acquires the form

$$u(x, t) = 2k^2\,\mathrm{ch}^{-2}\left\{k(x - 4k^2 t) + \chi_0\right\} . \tag{10}$$

Expression (10) corresponds to the solitary wave observed by Russell in 1834.

Solution (10) of the Korteveg–de Vries equation represents a running wave. This means that it depends on the coordinate x and the time t in terms of the variable $\xi = x - c_0 t$ characterizing the position of a coordinate point moving with the wave velocity c_0.

Thus, unlike the simple wave equation, the Korteveg–de Vries equation has, as a solution, a wave propagating only in one direction. However, owing to the additional terms uu_x and u_{xxx} it accounts for more complicated effects. The Korteveg–de Vries equation is also approximate since in its derivation the smallness property of the parameters ε and δ is used. With neglect of the influence of these parameters, making them vanish, we obtain one of the parts of the d'Alembert solution. If we account for the influence of the parameters ε and δ more exactly, we arrive at a more complicated equation than (5) with higher-order derivatives. Therefore, the solution of the Korteveg–de Vries equation is valid for describing waves only at some distance from the site of wave formation and for a fixed time interval. In this sense the Korteveg–de Vries equation should be treated as a certain approximation (a mathematical model) corresponding, with a high degree of accuracy, to the true process of wave propagation on water.

It is easy to verify that the principle of superposition of solutions does not hold for the Korteveg–de Vries equation and therefore it is nonlinear and describes nonlinear waves.

Korteveg–de Vries Solitons. At present, it is surprising that Russell's discovery even after confirmation by Korteveg and de Vries has not had repercussions in science. Furthermore, Korteveg, one of the authors of the equation, lived a long life and was a famous scientist. But when in 1945 the scientific community celebrated his 100th birtday, his work carried out with de Vries was not mentioned in the list of his best publications. Those who had prepared this list considered this work not deserving of attention. Only a quarter of a century later precisely this work has been recognized as the main scientific achievement of Korteveg.

However, this indifference to Russell's solitary wave becomes becomes understandable if we take into consideration the special features of this discovery. At that time the physical world seemed to be linear and the principle of superposition was considered to be one of the fundamental principles of most physical theories. Therefore, scientists did not attach importance to the discovery of an exotic wave on water.

Reverting to the discovery of the solitary wave happened by chance, and at first it seemed not to have any relation to the latter. We are indebted to Fermi, who in 1952 asked Ulam and Pasta, two young physicists, to solve a nonlinear problem on a computer. They had to calculate vibrations of identical loads connected to each other by little springs, which on deviation from the equilibrium position by Δl acquired the restoring force $p\Delta l + p_1 \Delta l^2$. In this case, the nonlinear addend was assumed to be small compared to the main force $p\Delta l$. Creating the initial deviation, the investigators wanted to trace how the mode will be distributed with respect to all the others. Upon calculating this problem, they failed to obtain the expected result; however, they found that in the initial stage the energy was actually transferred to two or three modes but then the initial state was regained. This paradox, called the Fermi–Pasta–Ulam paradox, became known to some mathematicians and physicists. In particular, Kruskal

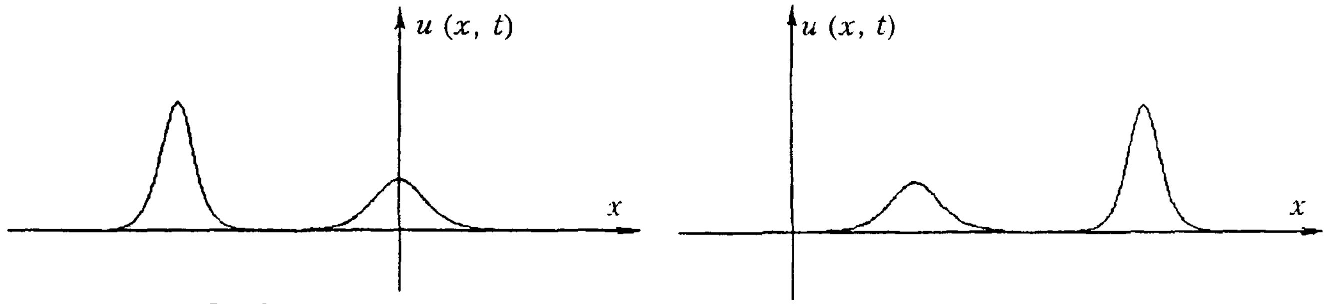

Fig. 2. Two solitons described by the Korteveg–de Vries equation before (at the left) and after (at the right) their interaction.

and Zabusky, two American physicists, came to know about this problem and decided to continue computational experiments with the model suggested by Fermi.

The problem of Fermi, Pasta, and Ulam consisted in solving the system of ordinary differential equations

$$m \frac{d^2 y_i}{dt^2} = F_{i,i+1} - F_{i-1,i} \quad (i = 1, \dots, 63), \tag{11}$$

where

$$F_{i-1,i} = p\Delta l + p_1 \Delta l^2, \quad \Delta l = y_i - y_{i-1} \tag{12}$$

for the corresponding boundary and initial conditions.

Kruskal and Zabusky established that with decrease in the distance between the loads and with unbounded increase in their number the equation used by Fermi, Pasta, and Ulam turns into the Korteveg–de Vries equation, i.e., in essence, the problem suggested by Fermi reduces to a numerical solution of the Korteveg–de Vries equation for the Russell solitary wave. At almost the same time it was shown that the Korteveg–de Vries equation is encountered in describing ionic-sonic waves in a plasma. Then it became clear that this equation is applicable to many fields in physics and, consequently, the solitary wave described by it represents a well-known phenomenon.

Going on with computational experiments on modeling the propagation of such waves, Kruskal and Zabusky considered their collision. We dwell on discussion of this important fact. Let two solitary waves described by the Korteveg–de Vries equation exist that differ in their amplitudes and move one after the other (Fig. 2). From formula (9) for solitary waves it follows that the larger their velocity, the higher their amplitude, while the wave width decreases with an increase in the amplitude. Thus, high solitary waves move faster. The wave with the larger amplitude will overtake the wave with the smaller amplitude moving in front of it. Next, the two waves will continue their motion as a single whole, and then they separate. A remarkable feature of these waves is the fact that after their interaction they retain their shape and velocity. After the collision the two waves are only shifted by some distance.

The process in which the shape and the velocity are retained after interaction of the waves resembles elastic collision of particles. Therefore, Kruskal and Zabusky called such waves solitons (which originates from the word "solitary") [6]. This special name for solitary waves, which keeps tune with the electron, the proton, and many other elementary particles, is now generally accepted.

The solitary waves discovered by Russell behave, indeed, like particles. It has turned out that a large wave does not pass through a small one in their interaction. When solitary waves come into contact, the large wave slows down and decreases to the dimensions of the small wave, the solitons separate, and the large soliton runs forward. Thus, solitons behave like elastic tennis balls. By a soliton is meant a solitary nonlinear wave that retains its shape and velocity when it moves and when it collides with similar solitary waves, i.e., it represents a stable formation. The only possible result of the interaction of solitons is some shift of their phases.

Method of the Inverse Scattering Problem. After the soliton discovery some attempts were made to find transformations to simplify the Korteveg–de Vries equation. Such attempts have failed. However, Miura has found a transformation (which is now called the Miura transformation) in the form

$$u = v_x - v^2 , \tag{13}$$

that relates a solution of the Korteveg–de Vries equation (5) and a solution of the modified Korteveg–de Vries equation

$$v_t - 6v^2 v_x + v_{xxx} = 0 \tag{14}$$

by virtue of the relation

$$u_t + 6uu_x + u_{xxx} = \left(\frac{\partial}{\partial x} - 2v\right)(v_t - 6v^2 v_x + v_{xxx}) . \tag{15}$$

Miura transformation (13) as such does not simplify at all the solution of the Korteveg–de Vries equation since it relates the solutions of two nonlinear equations. Nevertheless, precisely this transformation has served as a key to finding a method to solve the Cauchy problem for the Korteveg–Vries equation.

Since the Korteveg–de Vries equation permits the use of the Galilean group of transformations, then some number λ can be introduced in (13). As a result, (13) and (14) acquire the form of the system of equations

$$v_x = u + \lambda + v^2 , \tag{16}$$

$$v_t = -\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + 2v\right)(u - 2\lambda)\right] . \tag{17}$$

If we perform the substitution

$$v = -\frac{\Psi_x}{\Psi} , \tag{18}$$

then the system of equations (16), (17) can be represented in the form

$$\Psi_{xx} + (u + \lambda)\Psi = 0 , \tag{19}$$

$$\Psi_T = (d(t) + u_x)\Psi - 2(u - 2\lambda)\Psi_x , \tag{20}$$

where $d(t)$ is the variable in integration.

The system of equations (19), (20) is equivalent to the Korteveg–de Vries equation since using the consistency condition

$$(\Psi_{xx})_t = (\Psi_t)_{xx} , \tag{21}$$

we obtain the Korteveg–de Vries equation. This linear system of equations relative to the new function $\Psi(x, t, \lambda)$ is called the Lax pair, since in 1968 Lax showed that a whole family of nonlinear differential equations can be represented in a similar form [7]. It is evident that the first equation of the Lax pair coincides with the steady-state Schrödinger equation. The system of equations (19), (20) is used to solve the Cauchy problem of the Korteveg–de Vries equation by the method of the inverse scattering problem.

Given the initial condition

$$u(x, t = 0) = \varphi(x) \tag{22}$$

it is required that a solution of the Korteveg–de Vries equation be found.

If $\varphi(x)$ satisfies the condition

$$\int_{-\infty}^{\infty} (1 + x^2)\, |\varphi(x)|\, dx < \infty\,, \tag{23}$$

then from the solution of the direct scattering problem (19) scattering data are determined from the prescribed condition $\varphi(x)$:

$$S = \left\{ r(k, 0), \chi_n(0), |b_n(0)|, \quad n = 1, \ldots, N \right\}, \tag{24}$$

where $\chi_n(0)$ $(\lambda_n = \chi_n^2)$ are zeros on the imaginary axis of the complex plane λ (energy levels of bound states).

Next, from the second equation of the Lax pair the time dependence of the scattering data is found in the form [8]

$$\begin{gathered} \chi_n(t) = \chi_n(0) \quad (n = 1, \ldots, N)\,, \quad b_n(t) = b_n(0) \exp\left\{4\chi_n^3 t\right\}, \\ r(k, t) = r(k, 0) \exp\left\{8ik^3 t\right\}. \end{gathered} \tag{25}$$

On the other hand, with scattering data (24) being known, the potential in the steady Schrödinger equation can be recovered from the solution of the Gel'fand–Levitan–Marchenko integral equation

$$K(x, y) + B(x + y) + \int_x^{\infty} B(y + z)\, K(x, z)\, dz = 0\,, \tag{26}$$

where the function $B(\xi)$ is determined from the scattering data:

$$B(\xi) = \sum_{n=1}^{N} b_n^2 \exp(-\chi_n \xi) + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) \exp(ik\xi)\, dk\,. \tag{27}$$

Therefore, substituting (25) into (27) and then solving Eq. (26), we obtain the function $K(x, y, t)$, which at $y = x$ with the use of the formula

$$u(x, t) = -2 \frac{\partial}{\partial x} K(x, x, t) \tag{28}$$

gives the solution of the Cauchy problem for the Korteveg–de Vries equation. This scheme represents, in essence, the method of the inverse scattering problem, which was developed [8] by Gardner, Greene, Kruskal, and Miura in 1967.

Thus, it has turned out that the Cauchy problem for the Korteveg–de Vries equation is solved as a sequence of linear problems, although the Korteveg–de Vries equation itself is, of course, nonlinear. The method elaborated by Gardner, Greene, Kruskal, and Miura has given a powerful incentive for investigation of nonlinear equations in mathematical physics and has made a good start to numerous remarkable achievements in this field.

Group and Topological Solitons. In practice, waves propagate, as a rule, in groups. This is due to the fact that it is difficult to form a monochromatic wave. Similar groups of waves on water have been observed from time immemorial. However, only in 1967 could Benjamin and Feir answer the question of why a typical picture on water is "flocks" of waves. They showed that a simple periodic wave on deep water is unstable (now this phenomenon is called the Benjamin–Feir instability) and due to the instability waves on water are separated into groups. An equation to describe propagation of groups of waves on water was obtained by V. E. Zakharov in 1968. It is called the nonlinear Schrödinger equation and is as follows:

$$q_t = iq_{xx} \pm 2iq^2 q^*\,. \tag{29}$$

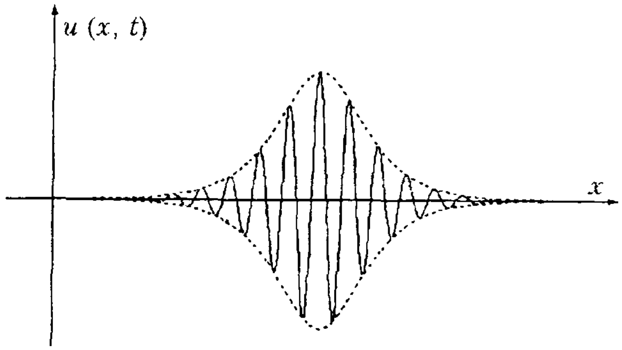

Fig. 3. Approximate form of a group soliton (the dashed line).

In 1971 N. E. Zakharov and A. B. Shabat [10] showed that this nonlinear equation also has solutions in the form of solitons; moreover the Cauchy problem for the nonlinear Schrödinger equation, just as for the Korteveg–de Vries equation, can be solved by the method of the inverse scattering problem. The solitons of the nonlinear Schrödinger equation differ from the Korteveg–de Vries solitons discussed above in their correspondence to the shape of the envelope of a group of waves. Outwardly they resemble modulated radio waves and are called group solitons and, sometimes, envelope solitons. This term reflects conservation of the envelope of a wave package on interaction (an analog of the dashed line shown in Fig. 3), although the waves beneath the envelope move with a velocity that differs from the group velocity. Here, the shape of the envelope is described by the relation

$$q\,(x,\,t) = q_0 \,\mathrm{ch}^{-1} \left(\frac{x - c_0 t}{l} \right), \tag{30}$$

where q_0 is the wave amplitude.

Commonly, beneath the envelope 14 to 20 waves occur, of which the middle wave is the largest. This is associated with the well-known fact that the highest wave in a group on water is between the seventh and tenth waves ("the ninth roll"). If a group of waves forms from a larger number of waves, it breaks down into several groups.

The nonlinear Schrödinger equation is also widely used to describe waves in different fields of physics. It was suggested by Schrödinger in 1926 to analyze the fundamental properties of quantum systems and was initially used to describe interactions of intraatomic particles. The generalized or nonlinear Schrödinger equation describes a whole complex of phenomena in the physics of wave processes and is used to describe the evolution of the envelope of a wave packet in many physical systems.

The solitons described by the Korteveg–de Vries equation and the nonlinear Schrödinger equation do not exhaust the diversity of these remarkable nonlinear objects. A soliton no less popular than those described above is the so-called topological soliton, which also has its own interesting history and an extensive sphere of applications [2-4]. It appears in all processes described by a nonlinear equation of the form

$$v_{xt} = \sin v\,. \tag{31}$$

It first appeared in the last century in the Lobachevskii geometry to describe surfaces of negative curvature and at present is called the sine-Gordon equation.

As far back as in the end of the XIXth century, Backlund showed that (31) has special transformations (now they are known as Backlund transformations) that make it possible to find its analytical solutions successively.

In 1962, in analyzing the interaction of elementary particles the English physicists Perring and Skirme carried out numerical calculations with the use of (31) [4]. According to their calculations solitary waves that are solutions of the sine-Gordon equation did not change their properties after interaction. Their work preceded the computational experiment with the Korteveg–de Vries solitary wave by three years. However, Perring and Skirme did not introduce the notion of a soliton.

Theory of Solitons and Nonlinear Mathematical Physics. Since the time of the appearance of differential and integral calculus many mathematicians have reflected upon the question of what a solution of a differential equation is.

Already at the beginning of the XIXth century scientists found that no combination of the functions known at that time can express the unknown dependence in a differential equation. Then the idea of broadening the class of mathematical functions by means of which solutions of differential equations can be expressed emerged. However, investigators came up against a number of difficulties. This circumstance led to the idea of investigating solutions of differential equations using the equations themselves, since from the geometric point of view their solutions represent some line in a plane, i.e., an integral curve. This approach is characteristic of the qualitative theory of differential equations.

However, Cauchy paid attention to the fact that it was convenient to consider solutions of differential equations as functions of a complex variable. Precisely this viewpoint lies behind investigations of solutions in the analytical theory of differential equations [11, 12].

An extremely important notion in the theory of functions of a complex variable is the notion of an analytic function.

Definition 1. A function $f(z)$ is said to be analytic at a given point if it is differentiable not only at the given point but also in some neighborhood of this point.

Definition 2. A function that is analytic at all points of some domain is said to be analytic in this domain.

For instance, for the first-order equation

$$w_z = F(w, z) , \tag{32}$$

considered in the complex plane, it is assumed that w and z are complex variables and F is an nalytic function of the variable z and a rational function of the variable w.

For Eq. (32), in the analytical theory of differential equations a solution is sought that takes the initial value $w = w_0$ at $z = z_0$, where z_0 and w_0 are two specified complex numbers. Theorems of existence and uniqueness that are extended to an equation in a complex variable determine its solution inside some circle and specify an element of the analytic function, and if it satisfies the differential equation, the latter will be satisfied by analytic continuations of the element to the entire domain. Therefore the analytic function as a whole is also a solution of the same differential equation.

Definition 3. The points of the plane, at which the single-valued function $f(z)$ is analytic are said to be regular points of this function, while the points at which the function $f(z)$ is not analytic (in particular, the points at which $f(z)$ is not defined) are referred to as singular points.

Investigation of the behavior of solutions in the vicinity of singular points is an important problem. And although the latter is local, it is closely connected with investigation of the behavior of a solution as a whole.

A general classification of singular points of arbitrary analytic functions (not necessarily solutions of differential equations) has been given by Painlevé. It is based on the number of values of the function taken by it in going around the singular point analyzed.

Definition 4. If a function changes its value in going around a singular point, this singular point is called critical. If in going around the point the value of the function remains unchanged, the singular point is called noncritical.

An example of a critical singular point is the point $z = 0$ for the functions $w = \sqrt{z}$.

Painlevé also singled out a class of algebraic singular points to which both critical and noncritical poles are referred. For instance, the point $z = 0$ is a noncritical pole for the function $w = 1/z$ and a critical pole for the function $w = 1/\sqrt{z}$.

Fuchs noted that solutions of differential equations can have singular points that do not depend on the initial conditions. In this connection, he subdivided all singular points of solutions of differential equations into moving and fixed points.

Definition 5. Singular points of solutions of differential equations whose position does not depend on the initial data determining the solution are called fixed singular points.

Definition 6. Singular points of solutions of differential equations whose position depends on the initial data are called moving singular points.

For instance, the solution of the problem

$$v_t = -kv^2, \quad v(t = t_0) = v_0, \tag{33}$$

has the form

$$v = \frac{v_0}{v_0 k (t - t_0) + 1}. \tag{34}$$

The singular point of solution (34) is the pole $t^* = t_0 - (kv_0)^{-1}$, whose position depends on the initial data t_0 and v_0, and therefore the solution of Eq. (33) has noncritical moving singular points.

Solutions of differential equations can have both critical and noncritical moving singular points. Among all singularities (if any) of solutions of differential equations four different versions can be encountered: 1) the solution has neither critical nor moving singular points; 2) the solution has fixed critical singular points; 3) the solution has moving noncritical singular points; 4) the solution has movable critical singular points.

In the analytical theory of differential equations it is proved that solutions of linear equations can have only fixed critical singular points; moreover, all critical singular points of solutions are determined by the singular points of the coefficients of the differential equation itself. Thus, the singularities of solutions of such differential equations pertain to the first and second versions mentioned above.

However, in the case of a nonlinear differential equation, the solutions can have both moving and fixed critical singular points.

In 1884 Fuchs and Poincaré formulated the problem of seeking nonlinear differential equations that have fixed critical singular points. Actually, they formulated the problem of finding nonlinear differential equations that have analytical solutions. In so doing, new functions as solutions of nonlinear differential equations can be determined.

In the same year Fuchs proved the theorem that among all first-order equations of the form (32) with a function F that is rational with respect to w and locally analytic with respect to z only the Riccati equation

$$w_z = P_0(z) + P_1(z)\, w + P_2(z)\, w^2 \tag{35}$$

has no moving critical singular points.

S. A. Kovalevskaya, who knew about the results obtained by Fuchs, took the following important step in the analytical theory of differential equations in solving the problem of the motion of a solid body with a fixed point in the gravitational field (the top problem). She proved that solutions of the problem under consideration did not have moving critical singular points only for three sets of values of the problem parameters. Solutions of the problem in first two cases were known from the works of Euler and Lagrange, while for the third case Kovalevskaya found new solutions and became the first investigator to reveal advantages in solving a differential equation when its solution had no moving critical singular points. In 1888, Kovalevskaya was awarded the Bordin prize of the French Academy of Sciences for a valuable contribution to the solution of the problem of the rotation of a solid body.

A short time later, Painlevé began an investigation of the second-order differential equations

$$w_{zz} = F(z, w, w_z), \tag{36}$$

where the function F is rational with respect to w and w_z and locally analytic with respect to z.

Together with Gambier, Garnier, and other followers Painlevé showed that among all possible second-order nonlinear equations of the form (36) the solutions of only 50 canonical equations have no moving critical singular points. The solutions of 44 equations of these fifty equations can be expressed in terms of elementary or well-known special functions, and for solutions of the remaining six equations Painlevé and Gambier introduced new special functions now called Painlevé transcendental functions. Thus, Painlevé and his followers succeeded in finding six new nonclassical functions determined by solutions of nonlinear differential equations of second order. The last of these functions (more exactly, the nonlinear differential equation determining this function) was found in 1906.

In the works of Painlevé and his followers (in the wake of Kovalevskaya's work) it was established that if a solution of a differential equation has no critical moving singular points, then the general solution of this differential equation can be obtained. This property of differential equations is now called Painlevé's property. We can give the following its definition of it.

Definition 7. An ordinary differential equation is considered to possess Painlevé's property if the general solution of this equation has no critical moving singular points.

An investigation of differential equations for Painlevé's property is called Painlevé analysis of differential equations. There are several methods that allow such an analysis to be carried out. However, we have no possibility of dwelling on a discussion of these issues.

Painlevé's property for an ordinary differential equation is essentially a criterion for the existence of the general solution of the differential equation. If the latter possesses Painlevé's property its general solution can be obtained, and if it does not possess Painlevé's property, the general solution cannot, as a rule, be obtained. However, there are cases where the initial equation does not satisfy Painlevé's propery but after substitution it reduces to an equation possessing this property, and therefore the initial equation also has a solution in the form of a formula.

In recent years interest in investigation of differential equations for Painlevé's property has grown drastically since a close relation of it to nonlinear partial differential equations solved by the method of the inverse scattering problem has been found.

After the Zakharov–Shabat method was generalized by Ablowitz, Kaup, Newell, and Segur in 1974, who established that the method of the inverse scattering problem was applicable for solution of many nonlinear partial differential equations [2], the question of a criterion by means of which one could establish whether a nonlinear equation has soliton solutions or not arose.

In this connection, Ablowitz and Segur turned their attention to the fact that if nonlinear partial differential equations having solutions in the form of solitons are transformed into ordinary differential equations, one obtains equations possessing Painlevé's property. In 1980, these authors together with Ramani formulated a test for checking the existence of soliton solutions for nonlinear partial differential equations [13, 14]. Essentially, this test consists in the following: if by means of transformations a nonlinear partial differential equation reduces to ordinary differential equations with Painlevé's property, then the method of the inverse scattering problem is applicable to this a nonlinear partial differential equation and it has soliton solutions.

However, in practice this test turned out to be inconvenient and in 1983 Weiss, Tabor, and Carnevalle suggested an extension of Painlevé's property to nonlinear partial differential equations called the method of singular varieties [15].

Let a nonlinear partial differential equation be given in the general form

$$E(u, u_x, u_t, \dots, x, t) = 0, \tag{37}$$

then the solution of the nonlinear partial differential equation in this method is represented as the sum

$$u(x, t) = \sum_{j=0}^{M} u_j(x, t)\, z(x, t)^{j-n}, \tag{38}$$

where $z(x, t)$ is a new function; u_j depends on the derivatives of the function $z(x, t)$ with respect to x and t; n is the smallest power of $z(x, t)$ obtained after substitation of (38) into the leading terms of the initial equation (37); M is a power that is commonly assumed to be equal to n.

It turned out that the representation (38) of the solution of the initial equation (37) leads to a systematic procedure for finding the Lax pairs (if any) for Eq. (37) [15-20]. Moreover, this method makes it possible to find particular analytical solutions of the initial equation, which is not solved as a whole by the method of the inverse scattering problem [21-27]. Recently it has been shown that the representation of the solution of (37) is, in essence, a transformation that maps one equation solved by the inverse-problem method on another equation solved by this method [28, 29]. These transformations can be used in constructing families of exactly or partially solved nonlinear equations of mathematical physics [30, 31] and lead to iteration formulas for constructing rational soliton solutions of nonlinear equations solved by the method of the inverse scattering problem [32, 33].

The linkage between nonlinear partial differential equations having soliton solutions and ordinary differential equations possessing Painlevé's property has made it possible to progress in solving the Fuchs and Poincaré problem of seeking new functions determined by solutions of nonlinear equations. Recently in [34, 35] nonlinear ordinary differential equations of fourth order have been suggested that possess properties similar to Painlevé transcendental functions. It seems that these equations determine, as Painlevé transcendental functions do, new nonclassical functions [36-38].

Solitons are being studied intensely at present. This is attributed to the fact that such phenomena exist in nature and can find wide application in engineering.

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NOTATION

$u(x, t)$, wave characteristic in Eqs. (1), (5); c_0, wave velocity; x, coordinate; t, time; f and q, functions in the solution of wave equation (1); C, wave velocity on water; g, free-fall acceleration; a, wave amplitude; h, depth of the body of water; ε and δ, dimensionless small parameters; α, β, and γ, real roots of cubic equation (7); ξ, coordinate of the running wave; $cn(x, t)$, Jacobi elliptic function; T, period of the cnoidal wave; C_0, C_1, C_2, k, and χ_0, arbitrary constants; $K(s)$, complete elliptical integral of the 1st kind; y_i, deviation of the i-th mass from the position of equilibrium, $i = 1, ..., 63$; m, particle mass in the Fermi–Pasta–Ulam model; $F_{i,i+1}$, force acting from the side of the i+1-th mass to the i-th mass; l, wavelength; Δl, difference of shifts from the position of equilibrium; p and α, parameters; $v(x, t)$, wave characteristic described by Eqs. (14), (31); λ, constant number in the system of equations (16), (17); $\Psi(x, t)$, wave function in the system of equations (19), (20); $\varphi(x)$, initial condition for the Korteveg–de Vries equation; $r(k, 0)$, reflection coefficient; $\chi_n(0)$, zeros on the imaginary axis; $b_n(0)$, normalization constants for eigenfunctions; k, value of the momentum in the Schrödinger equation; $K(x, y)$, kernel in the Gel'fand–Levitan–Marchenko equation; $B(x + y)$, function determined in terms of the scattering data; $\mathrm{ch}(x, t)$, hyperbolic cosine; q_0, constant; q^*, complex conjugate of q; w and z, complex variables; $F(w, z)$, function rational relative to w and analytic relative to z in Eqs. (32), (35), and (36); E, designation of the partial differential equation (37); $u_j(x, t)$ and $z(x, t)$, coefficient and new function in the method of Weiss, Tabor, and Carnevalle; U, unknown quantity in the cubic equation; ξ_0, arbitrary constant.

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